Effective SO Superpotential for $\mathcal{N} = 1$ Theory with N_f Fundamental Matter

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Abstract

Motivated by the duality conjecture of Dijkgraaf and Vafa between supersymmetric gauge theories and matrix models, we derive the effective superpotential of $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group $SO(N_c)$ and arbitrary tree level polynomial superpotential of one chiral superfield in the adjoint representation and N_f fundamental matter multiplets. For a special point in the classical vacuum where the gauge group is unbroken, we show that the effective superpotential matches with that obtained from the geometric engineering approach.

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1 Introduction

Large N topological duality relating $U(N)$ Chern-Simons gauge theory on S^3 to A-model topological string [1] and its embedding in the superstring context [2] has led to interesting interconnections between geometry of Calabi-Yau three-folds (CY_3) and $\mathcal{N}=1$ supersymmetric gauge theories. Strong coupling dynamics of supersymmetric gauge theories can be studied within the superstring duality [2] by geometrically engineering D-branes. Using the geometric considerations of dualities in IIB string theory, Cachazo et al [3] have obtained low energy effective superpotential for a class of CY_3 geometries whose singular limit is given by

$$W'(x)^2 + y^2 + z^2 + w^2 = 0 \ , \tag{1.1}$$

where $W(x)$ is a polynomial of degree $n+1$. In fact, the low energy effective superpotential corresponds to a $\mathcal{N}=1$ supersymmetric $U(N)$ Yang-Mills with adjoint scalar Φ and tree level superpotential $W_{tree}(\Phi) = \sum_{k=1}^{n+1}(g_k/k)Tr\Phi^k$.

The mirror version of the large N topological duality conjecture [1] was considered in ref. [4] relating topological B strings on the CY_3 geometries [3] to matrix models. The potential of the matrix model $W(\Phi) = (1/g_s)W_{tree}(\Phi)$ where Φ denotes a hermitian matrix. Further, Dijkgraaf-Vafa have conjectured that the low-energy effective superpotential can be obtained from the planar limit of these matrix models [4–6]. The Dijkgraaf-Vafa conjecture was later on proved by various methods: (i) by factorization of Seiberg-Witten curves [7], (ii) using perturbative field theory arguments [8] and (iii) generalized Konishi anomaly approach [9].

The extension of topological string duality relating Chern-Simons theory with SO/Sp gauge groups to A-model closed string on an orientifold of the resolved conifold was studied by Sinha-Vafa [10]. Generalizing the geometric procedure considered for $U(N)$ [3], the effective superpotential for $\mathcal{N}=1$ supersymmetric theories with SO/Sp gauge groups with $W_{tree} = \sum_k (g_{2k}/2k)Tr\Phi^{2k}$ where Φ is adjoint scalar superfield were derived for the orientifolds of the CY_3 geometries [11]. These effective superpotentials have also been computed within perturbative gauge theory [12], using matrix model techniques in [13] and using the factorization property of $\mathcal{N}=2$ Seiberg-Witten curves [14].

Related works involving second rank tensor matter fields have been considered in refs. [15–17]. The geometric engineering of $\mathcal{N}=1$ gauge theories with unitary gauge group and matter in the adjoint and symmetric or antisymmetric representations has been investigated in [18]. Also for the SO/Sp theory with symmetric/antisymmetric tensor, the geometric construction was studied in [19].

So far, the effective superpotential computation involved $\mathcal{N}=1$ supersymmetric gauge theories with either adjoint matter or second rank tensor matter. The inclusion of matter transforming in the fundamental representation of these gauge groups can also be studied within the Dijkgraaf-Vafa setup [20–26]. For $U(N)$ gauge group, it was shown that the effective superpotential gets contributions from the matrix model planar diagrams with zero or one boundary [20,22]. In [24], it has been shown

that the $U(N)$ effective superpotential for theories with N_f fundamental flavors can be calculated in terms of quantities computed in the pure gauge theory. Chiral $\mathcal{N}=1$ $U(N)$ gauge theories with antisymmetric, conjugate symmetric, adjoint and fundamental matter have also been studied in Refs. [19, 27].

Further, geometric engineering of the supersymmetric theories with N_f fundamental flavors was considered in [28, 29] by placing $D5$ branes at locations given by the the mass m_a $(a = 1, 2, \ldots N_f)$ which are not the zeros of $W'(x) = 0$. The SO/Sp effective superpotential computations from matrix model approach have been presented for tree level superpotential of adjoint matter upto quartic terms [25, 26]. Though there are several indications as to how the $U(N)$ effective superpotential can be related to SO/Sp groups, it is certainly not a proof. So we need to explicitly obtain the results using an independent method as elaborated in this paper. We consider $\mathcal{N}=1$ supersymmetric $SO(N_c)$ gauge theory with arbitrary tree level superpotential of one chiral superfield in the adjoint representation and N_f fundamental matter multiplets. We use the technique developed in [24] to calculate the effective superpotential of this theory.

The organization of the paper as follows: In section 2, we briefly discuss the relevant matrix model and its free energy. Then we discuss the $SO(N_c)$ effective superpotential for $\mathcal{N}=1$ supersymmetric theory with fundamental matter in section 3. In particular, for a point in classical vacuum where the gauge group is unbroken, we obtain a neat expression for the effective superpotential for a most general tree level superpotential involving one adjoint matter field. We also give a formal expression for a generic vacua where the gauge group is broken. In section 4, we recapitulate the geometric considerations of dualities. Then, we evaluate the effective superpotential for an example involving sixth power tree level potential and show that the results agree with our expressions in section 3 for the unbroken case. We conclude with summary and discussions in section 5.

2 Relevant Matrix Model

Let us consider $\mathcal{N}=1$ supersymmetric $SO(N_c)$ gauge theory with one adjoint field Φ and N_f flavors of quarks Q^I's with mass m_I's $(I = 1, 2, \ldots N_f)$ in the vector(fundamental) representation. The tree level superpotential of this theory is given by [25]

$$W_{tree} = W(\Phi) + \sum_{I=1}^{N_f} \left(\tilde{Q}\Phi Q + m_I \tilde{Q} Q \right) \ , \tag{2.1}$$

where $W(\Phi)$ is a polynomial with even powers of Φ:

$$W(\Phi) = \sum_{k=1}^{n+1} \frac{g_{2k}}{2k} Tr \Phi^{2k} \, . \tag{2.2}$$

According to Dijkgraaf-Vafa conjecture, the effective superpotential of this theory can be obtained from the planar limit of the matrix model whose tree level potential

is proportional to W_{tree}. Hence the partition function of the matrix model is [25]

$$Z = e^{-\mathcal{F}} = \int \mathcal{D}\Phi \mathcal{D}Q \; exp\left\{ -\frac{1}{g_s}\left[W(\Phi) + \sum_{I=1}^{N_f} \left(\tilde{Q}\Phi Q + m_I \tilde{Q} Q \right) \right] \right\} , \qquad (2.3)$$

where Φ is $M \times M$ real antisymmetric matrix and Q, $\tilde{Q}$ are M dimensional vectors. For this theory, the Dijkgraaf-Vafa conjecture can be generalized [30] to obtain effective superpotential as a function of glueball field $S = TrW_\alpha W^\alpha$:

$$W_{eff} = N_c \frac{\partial \mathcal{F}_{s^2}}{\partial S} + 4\mathcal{F}_{RP^2} + \mathcal{F}_{D^2} \, , \qquad (2.4)$$

where $\mathcal{F}_{s^2}$ is a free energy of a diagram with topology of sphere, $\mathcal{F}_{RP^2}$ is free energy of the diagram with cross-cap one (topology of RP^2) and $\mathcal{F}_{D^2}$ is the free energy of the diagram with one boundary (topology of D^2). It is also known that [12]

$$\mathcal{F}_{RP^2} = -\frac{1}{2}\frac{\partial \mathcal{F}_{s^2}}{\partial S} \, . \qquad (2.5)$$

Using this, W_{eff} becomes

$$\begin{aligned} W_{eff} &= (N_c - 2)\frac{\partial \mathcal{F}_{s^2}}{\partial S} + \mathcal{F}_{D^2} \\ &= (N_c - 2)\frac{\partial \mathcal{F}_{\chi=2}}{\partial S} + \mathcal{F}_{\chi=1} \\ &= W_{VY} + (N_c - 2)\frac{\partial \mathcal{F}^{pert}_{\chi=2}}{\partial S} + \mathcal{F}_{\chi=1} \, , \end{aligned} \qquad (2.6)$$

where W_{VY} denotes the Veneziano-Yankielowicz potential [31]. Here we have absorbed $\mathcal{F}_{RP^2}$ in $\mathcal{F}_{\chi=2}$ and $\mathcal{F}_{\chi=1}$ contains the contribution to the free energy coming from the fundamental matter. As proposed by Dijkgraaf-Vafa, we need to take the planar limit of the matrix model. The planar limit can be obtained by taking $(M, N_f \to \infty)$ as well as $g_s \to 0$ such that $S = g_s M$ and $S_f = g_s N_f$ are finite. The total free energy of this matrix model can be expressed as an expansion in genus g and the number of quark loops h:

$$\mathcal{F} = \sum_{g,h} g_s^{2g-2} {S_f}^h \mathcal{F}_{g,h}(S) \; . \qquad (2.7)$$

Assuming that the fundamental quarks are massive compared to adjoint matter, we can integrate out the fundamental matter fields appearing quadratically in the partition function (2.3) to give

$$Z = e^{-\mathcal{F}} = \int \mathcal{D}\Phi \; exp\left[-\frac{1}{g_s} Tr\left(W(\Phi) + S_f \sum_{I=1}^{N_f} log(\Phi + m_I) \right) \right] \; . \qquad (2.8)$$

We are now in a position to calculate $\mathcal{F}_{\chi=1}$ and $\mathcal{F}_{\chi=2}$ contributions to the superpotential. In the following section we apply the method developed in [24] for the $SO(N_c)$ gauge theory with one adjoint matter field and N_f fundamental flavors.

3 Effective Superpotential

We wish to compute the exact effective superpotential of $\mathcal{N} = 2$ supersymmetric $SO(N_c)$ gauge theory with N_f flavors of quark loops in the fundamental representation, broken to $\mathcal{N} = 1$ by addition of a tree level superpotential $W(\Phi)$ given by eqn.(2.2). The supersymmetric vacua of the theory with superpotential (2.2) are obtained by diagonalizing Φ such that the eigenvalues are in the set of critical points of $W(\Phi)$ which are given by the zeros of

$$W'(x) = g_{2n+2}\; x \prod_{i=1}^{n} (x^2 + a_i^2). \tag{3.1}$$

Choosing all the N eigenvalues of $\langle\Phi\rangle = 0$ gives unbroken gauge group $SO(N)$. A generic gauge group $SO(N_0) \times \prod_{i=1}^{n} U(N_i)$ corresponds to N_0 eigenvalues of $\langle\Phi\rangle = 0$, N_1 eigenvalues of $\langle\Phi\rangle = ia_1, \ldots$. We will now look at the effective superpotential computation for both unbroken and broken gauge group in the next two subsections.

3.1 Unbroken gauge Group

Following the arguments in [24], for the effective superpotential evaluation we can still look at a point in the quantum moduli space of $\mathcal{N} = 2$ pure gauge theory where $r = [N_c/2]$ (rank of $SO(N_c)$) monopoles become massless [14]. This corresponds to the point where the Seiberg-Witten curve factorizes completely.

We have seen that the effective superpotential of this theory gets contributions from free energies $\mathcal{F}_{\chi=2}$ and $\mathcal{F}_{\chi=1}$ of the matrix model described in the previous section. We shall first calculate the contribution to the superpotential coming from $\mathcal{F}_{\chi=2}$ using the moduli associated with Seiberg-Witten factorization.

3.1.1 Contribution of $\mathcal{F}_{\chi=2}$

In this subsection we compute the contribution of $\mathcal{F}_{\chi=2}$ to the effective superpotential. This contains free energies of the diagrams having topology of S^2 and RP^2. Taking derivative of eqn.(2.7) with respect to g_s

$$\frac{\partial \mathcal{F}}{\partial g_s} = \sum_{g,h} g_s^{2g-3} (S_f)^h \left((2g-2)\mathcal{F}_{g,h} + S \frac{\partial \mathcal{F}_{g,h}}{\partial S} \right) + \sum_{g,h} h g_s^{2g-3} (S_f)^h \mathcal{F}_{g,h}(S)\,. \tag{3.2}$$

According to Dijkgraaf-Vafa, one should take the planar limit on the matrix model side. Also we take the number of quark loops, $h = 0$ for $\chi = 2$ free-energy computation. Planar limit of the above equation gives

$$\frac{\partial \mathcal{F}}{\partial g_s} = g_s^{-3} \left(S \frac{\partial \mathcal{F}_{\chi=2}}{\partial S} - 2\mathcal{F}_{\chi=2} \right)\,. \tag{3.3}$$

We can also differentiate eqn.(2.8) with respect to g_s to give

$$\frac{\partial \mathcal{F}}{\partial g_s} = -g_s^{-2}\langle TrW(\Phi)\rangle \,. \tag{3.4}$$

¿From the above two equations

$$g_s\langle TrW(\Phi)\rangle = 2\mathcal{F}_{\chi=2} - S\frac{\partial \mathcal{F}_{\chi=2}}{\partial S} \,. \tag{3.5}$$

The form of $W(\Phi)$ shows that the LHS contains the vacuum expectation values $\langle Tr\Phi^{2p}\rangle$. It is clear from the above equation that once we obtain the vevs $\langle Tr\Phi^{2p}\rangle$, we can easily compute $\mathcal{F}_{\chi=2}$. In the case of $\mathcal{N}=2$ $SO(N_c)$ gauge theory, the moduli are given by $u_{2p} = \frac{1}{2p}Tr\Phi^{2p}$. We are interested in the complete factorization of the Seiberg-Witten curve. The moduli that factorizes the Seiberg-Witten curve are given by [14]

$$\langle u_{2p}\rangle = \frac{N_c-2}{2p}C^p_{2p}\Lambda^{2p} \,, \tag{3.6}$$

where $C^i_j = \frac{j!}{i!(j-i)!}$ and Λ is the scale governing the running of the gauge coupling constant. The matrix model calculation of the vevs of the moduli done in the context of $SU(N_c)$ [32, 33] can be extended to $SO(N_c)$ giving

$$\langle u_{2p}\rangle = (N_c-2)\frac{\partial}{\partial S}\frac{g_s}{2p}\langle Tr\Phi^{2p}\rangle \,. \tag{3.7}$$

It is obvious from the above two equations that

$$\frac{\partial}{\partial S}g_s\langle Tr\Phi^{2p}\rangle = C^p_{2p}\Lambda^{2p} \,. \tag{3.8}$$

We denote the effective superpotential of pure $SO(N_c)$ gauge theory by W^0_{eff}. From eqn.(2.6) we can write

$$W^0_{eff} = (N_c-2)\frac{\partial \mathcal{F}_{\chi=2}}{\partial S} = \frac{(N_c-2)}{2}S\left(-log\frac{S}{\tilde{\Lambda}^3}+1\right) + (N_c-2)\frac{\partial \mathcal{F}^{pert}_{\chi=2}}{\partial S} \,. \tag{3.9}$$

The first term in the above equation is the Veneziano-Yankeilowicz superpotential [34] and $\tilde{\Lambda}^{3(N_c-2)}$ is the strong coupling scale of the $\mathcal{N}=1$ theory. The second term is perturbative in glueball superfield S with

$$\mathcal{F}^{pert}_{\chi=2} = \sum_{n\geq 1} f_n^{\chi=2}(g_{2p})S^{n+2} \,. \tag{3.10}$$

Once we compute the functions $f_n^{\chi=2}(g_{2p})$, we will have the effective superpotential of $SO(N_c)$ pure gauge theory. In order to compute these functions we need to take the derivative of (3.5) with respect to S

$$\frac{\partial}{\partial S}g_s\langle TrW(\Phi)\rangle = \frac{\partial \mathcal{F}_{\chi=2}}{\partial S} - S\frac{\partial^2 \mathcal{F}_{\chi=2}}{\partial S^2} \,. \tag{3.11}$$

Substituting eqns.(3.9,3.10) in the above equation, we get

$$(N_c-2)\frac{\partial}{\partial S}g_s\langle TrW(\Phi)\rangle = W^0_{eff} - S\frac{\partial W^0_{eff}}{\partial S}$$
$$= (N_c-2)\left[\frac{S}{2} - \sum_{n\geq 1} n(n+2)f_n^{\chi=2}(g_{2p})S^{n+1}\right] . \quad (3.12)$$

At the critical point of the superpotential, that is when $\partial W^0_{eff}/\partial S = 0$, we have

$$W^0_{eff} = (N_c-2)\frac{\partial}{\partial S}g_s\langle TrW(\Phi)\rangle = \sum_p g_{2p}\langle u_{2p}\rangle \, . \quad (3.13)$$

The glueball superfield can be obtained at the critical point by the following relation [14]:

$$S = \frac{\partial W^0_{eff}}{\partial log\Lambda^{N_c-2}} = \sum_{p\geq 1} g_{2p}C^p_{2p}\Lambda^{2p} \, . \quad (3.14)$$

Inserting eqn.(3.8) and eqn.(3.14) in eqn.(3.12) one gets

$$\sum_{p\geq 1}\frac{1}{2p}g_{2p}C^p_{2p}\Lambda^{2p} = \frac{1}{2}\sum_{p\geq 1} g_{2p}C^p_{2p}\Lambda^{2p} - \sum_{n\geq 1} n(n+2)f_n^{\chi=2}(g_{2p})S^{n+1} \, . \quad (3.15)$$

Substituting glueball field S in terms of Λ (3.14) and equating the powers of Λ on both sides of the above equation, we can extract the functions $f_n^{\chi=2}(g_{2p})$:

$$f_1^{\chi=2} = \frac{1}{8}\frac{g_4}{g_2^2}$$
$$f_{n\geq 2}^{\chi=2} = \frac{C^{n+1}_{2(n+1)}}{2^{n+2}(n+1)(n+2)}\frac{g_{2(n+1)}}{g_2^{n+1}} - \sum_{l=1}^{n-1}\frac{l(l+2)}{n(n+2)}f_l^{\chi=2}\sum_{\substack{p_1,\cdots p_{l+1}=1 \\ p_1+\ldots+p_{l+1}=n+1}}^{n+1}\frac{C^{p_1}_{2p_1}g_{2p}\ldots C^{p_{l+1}}_{2p_{l+1}}g_{2p_{l+1}}}{2^{n+1}g_2^{n+1}} \, , \quad (3.16)$$

Now that we have computed the functions $f_n^{\chi=2}(g_{2p})$, the $\chi=2$ contribution to the effective superpotential of the $SO(N_c)$ theory with one adjoint chiral superfield with arbitrary tree level superpotential is known exactly. From eqn.(3.9) and eqn.(3.10), it is given by

$$W^0_{eff} = (N_c-2)\left[\frac{S}{2}\left(-log\frac{S}{\tilde{\Lambda}^3}+1\right) + \sum_{n\geq 1}(n+2)f_n^{\chi=2}(g_{2p})S^{n+1}\right] \, . \quad (3.17)$$

In the case of quadratic tree level superpotential, that is when $g_{2p}=0$ for $p\geq 2$, the functions $f_n^{\chi=2}(g_{2p})$ vanish for all n. And we can fix the coupling scale $\tilde{\Lambda}^3$ to $2g_2\Lambda^2$ by the requirement that W^0_{eff} satisfies equation (3.14). The W^0_{eff} for the quartic

tree level superpotential can be obtained by substituting $g_{2p} = 0$ for $p \geq 3$ in the above result (3.17):

$$W^0_{eff} = W_{VY} + (N_c - 2)\left[\frac{3}{2}\left(\frac{g_4}{4g_2^2}\right)S^2 - \frac{9}{2}\left(\frac{g_4^2}{8g_2^4}\right)S^3 + \frac{45}{2}\left(\frac{g_4^3}{16g_2^6}\right)S^4 + \ldots\right] . \quad (3.18)$$

This is in perfect agreement with the result of [26] where it has been evaluated in terms of the matrix model as well as IIB closed string theory on Calabi-Yau with fluxes. Substitution of $g_{2p} = 0$ for $p \geq 4$ in eqn.(3.17) gives W^0_{eff} for the the theory with sixth order potential:

$$W^0_{eff} = W_{VY} + (N_c - 2)\left[\frac{3}{8}\frac{g_4}{g_2^2}S^2 + \left(\frac{5}{12}\frac{g_6}{g_2^3} - \frac{9}{16}\frac{g_4^2}{g_2^4}\right)S^3 + \left(-\frac{15}{8}\frac{g_4 g_6}{g_2^5} + \frac{45}{32}\frac{g_4^3}{g_2^6}\right)S^4 + \ldots\right] . \quad (3.19)$$

We now compare the result (3.17) with the corresponding result in the $SU(N_c)$ gauge theory with one adjoint matter. The effective superpotential of $SU(N_c)$ theory has been obtained in [24]. Comparison of the effective superpotentials of these two theories provides the following equivalence:

$$W^{0\,SO(N_c)}_{eff}(g_{2p}) = \frac{N_c - 2}{2N_c}\, W^{0\,SU(N_c)}_{eff}(g'_{2p} = 2g_{2p}) \, , \quad (3.20)$$

which agrees with the relation obtained in [14]. We shall now address the fundamental matter contribution $\mathcal{F}_{\chi=1}$ to the effective potential.

3.1.2 Contribution of $\mathcal{F}_{\chi=1}$

We differentiate the free energy given by eqn.(2.7) with respect to S_f

$$\frac{\partial \mathcal{F}}{\partial S_f} = \sum_{g,h} h g_s^{2g-2} (S_f)^{h-1} \mathcal{F}_{g,h}(S) \, . \quad (3.21)$$

We are interested in genus $g = 0$ and one quark loop $h = 1$ contribution in the planar limit $g_s \to 0$. The dominant term from eqn.(3.21) is $\partial_{S_f}\mathcal{F} = g_s^{-2}\mathcal{F}_{\chi=1}$. Differentiation of eqn.(2.8) with respect to S_f gives

$$\frac{\partial \mathcal{F}}{\partial S_f} = g_s^{-1} \sum_{I=1}^{N_f} \langle Tr\, log(\Phi + m_I)\rangle \, , \quad (3.22)$$

This implies

$$\mathcal{F}_{\chi=1} = g_s \sum_{I=1}^{N_f} \langle Tr\, log(\Phi + m_I)\rangle \, . \quad (3.23)$$

Expanding the above equation around the critical point $\Phi = 0$, we get

$$\mathcal{F}_{\chi=1} = \sum_{I=1}^{N_f} \left(S\, log\, m_I - \sum_{k=1}^{\infty} \frac{(-1)^k}{k m_I^k} g_s \langle Tr \Phi^k \rangle \right) \, . \quad (3.24)$$

Differentiating with respect to S and using eqn.(3.8) we get

$$\frac{\partial \mathcal{F}_{\chi=1}}{\partial S} = \sum_{I=1}^{N_f} \left(log\, m_I - \sum_{k=1}^{\infty} \frac{1}{2\, k m_I^{2k}} C_{2k}^{k} \Lambda^{2k} \right) . \tag{3.25}$$

Integrating the above equation with respect to S we obtain

$$\mathcal{F}_{\chi=1} = \sum_{I=1}^{N_f} S\, log\, m_I - \sum_{I=1}^{N_f} \sum_{k,l \geq 1} \frac{l g_{2l} C_{2l}^{l} C_{2k}^{k}}{2k(k+l) m_I^{2k}} \Lambda^{2(k+l)} + D \,, \tag{3.26}$$

where D is the constant of integration. We postulate

$$D = \sum_{I=1}^{N_f} W_{tree}(m_I) \,, \tag{3.27}$$

and we will see in the next section that the result agrees with the one obtained from Calabi-Yau geometry with fluxes. In order to write this expression in powers of S, we write $\mathcal{F}_{\chi=1}$ as

$$\mathcal{F}_{\chi=1} = S \sum_{I=1}^{N_f} log\, m_I + \sum_{n \geq 1} f_n^{\chi=1}(g_{2p}) S^{n+1} + \sum_{I=1}^{N_f} W_{tree}(m_I) \,. \tag{3.28}$$

Comparison with eqn.(3.26) gives the following recursive relation for the coefficients $f_n^{\chi=1}(g_{2p})$

$$\begin{aligned} f_1^{\chi=1} &= -\frac{1}{4} \sum_{I=1}^{N_f} \frac{1}{m_I^2 g_2} \\ f_{n\geq 1}^{\chi=1} &= -\frac{1}{2^{n+1} g_2^{n+1}} \left(\sum_{I=1}^{N_f} \sum_{k,l=1}^{n} \frac{l g_{2l} C_{2l}^{l} C_{2k}^{k}}{2k(n+1) m_I^{2k}} \right. \\ &\quad \left. + \sum_{q=1}^{n-1} f_q^{\chi=1} \sum_{\substack{p_1,\cdots p_{q+1} \\ p_1+\ldots+p_{q+1}=n+1}}^{n+1} C_{2p_1}^{p_1} g_{2p_1} \ldots C_{2p_{q+1}}^{p_{q+1}} g_{2p_{q+1}} \right) . \end{aligned} \tag{3.29}$$

The eqn.(3.28) alongwith eqn.(3.29) gives the effective superpotential from fundamental matter, for the most general W_{tree}. If we substitute $g_{2p} = 0$ for $p \geq 2$ in the above result, we get $\mathcal{F}_{\chi=1}$ for the gauge theory with quadratic superpotential. It is explicitly given by,

$$\mathcal{F}_{\chi=1} = \sum_{I=1}^{N_f} \left[S log\, m_I - \frac{1}{4} \frac{S^2}{{m_I}^2 g_2} - \frac{1}{8} \frac{S^3}{{m_I}^4 {g_2}^2} - \frac{5}{48} \frac{S^4}{{m_I}^6 {g_2}^3} - \ldots \right] + D \,. \tag{3.30}$$

Also substituting $g_{2p} = 0$ for $p \geq 3$, we get $\mathcal{F}_{\chi=1}$ for the theory with quartic superpotential.

$$\mathcal{F}_{\chi=1} = \sum_{I=1}^{N_f} \left[S log\, m_I + \left(-\frac{1}{4{m_I}^2 g_2} \right) S^2 + \left(-\frac{1}{8{m_I}^4 {g_2}^2} + \frac{g_4}{4{m_I}^2 {g_2}^3} \right) S^3 \right.$$
$$\left. + \left(-\frac{5}{48{m_I}^6 {g_2}^3} + \frac{9g_4}{32{m_I}^4 {g_2}^4} - \frac{9g_4^2}{16{m_I}^2 {g_2}^5} \right) S^4 + \ldots \right] + D\,. \tag{3.31}$$

If we set $g_{2p} = 0$ for $p \geq 4$, the resulting theory has sixth order tree level potential and the corresponding $\mathcal{F}_{\chi=1}$ is given by

$$\mathcal{F}_{\chi=1} = \sum_{I=1}^{N_f} \left[S log\, m_I + \left(-\frac{1}{4{m_I}^2 g_2} \right) S^2 + \left(-\frac{1}{8{m_I}^4 {g_2}^2} + \frac{g_4}{4{m_I}^2 {g_2}^3} \right) S^3 \right.$$
$$\left. + \left(-\frac{5}{48{m_I}^6 {g_2}^3} + \frac{9g_4}{32{m_I}^4 {g_2}^4} - \frac{9g_4^2}{16{m_I}^2 {g_2}^5} - \frac{15}{16}\frac{g_6}{m_I^2 g_2^4} \right) S^4 + \ldots \right] + D \tag{3.32}$$

The total effective superpotential of the theory under consideration is

$$\begin{aligned} W_{eff} &= W_{eff}^0 + \mathcal{F}_{\chi=1} \\ &= (N_c - 2) \left[\frac{S}{2} \left(-log \frac{S}{2g_2 \Lambda^2} + 1 \right) + \sum_{n\geq 1} (n+2) f_n^{\chi=2}(g_{2p}) S^{n+1} \right] \\ &+ S \sum_{I=1}^{N_f} log\, m_I + \sum_{n\geq 1} f_n^{\chi=1}(g_{2p}) S^{n+1} + \sum_{I=1}^{N_f} W_{tree}(m_I) \end{aligned} \tag{3.33}$$

3.2 Broken Gauge Group

In the previous subsection, we have computed the effective superpotential of $SO(N_c)$ supersymmetric gauge theory for unbroken gauge group. In this section we obtain the effective superpotential for broken gauge group. In particular we consider the following breaking pattern,

$$SO(N) \to SO(N_0) \times \prod_{i=1}^{n} U(N_i)\ , \tag{3.34}$$

such that $N = N_0 + 2\sum_{i=1}^{n} N_i$ and for every factor of the gauge group, there is a glueball superfield S_i. We introduce the variables, $e_0 = 0,\ e_i = ia_i,\ e_{-i} = -ia_i,\ i = 1, 2, \ldots, n$.

3.2.1 Contribution of $\mathcal{F}_{\chi=2}$

Let us first compute the free energy $\mathcal{F}_{\chi=2}$ for the pure gauge theory. For this case the eqn.(3.5), which has been used to evaluate $\mathcal{F}_{\chi=2}$ in the case of unbroken gauge

group, modifies to

$$g_s\langle TrW(\Phi)\rangle = 2\mathcal{F}_{\chi=2} - \sum_{i=-n}^{n} S_i \frac{\partial \mathcal{F}_{\chi=2}}{\partial S_i} \ . \tag{3.35}$$

The free energy $\mathcal{F}_{\chi=2}$ is a combination of non-perturbative part, coming from the Veneziano-Yankielowitz term [35] and a perturbative part:

$$\mathcal{F}_{\chi=2} = \sum_{i=-n}^{n} S_i W(e_i) - \frac{1}{4}\sum_{i=-n}^{n} S_i^2 log\left(\frac{S_i}{\alpha\Lambda\Delta_i}\right) - \frac{1}{2}\sum_{i,j=-n}^{n} S_i S_j log\left(\frac{e_i - e_j}{\Lambda}\right) + \sum_m \mathcal{F}^{(m)}_{\chi=2} \ , \tag{3.36}$$

where the perturbative part is contained in $\mathcal{F}^{(m)}_{\chi=2}$, which is polynomial of order m in S_i. Substitution of $\mathcal{F}_{\chi=2}$ given by eqn.(3.36) in eqn.(3.35) implies

$$g_s\langle TrW(\Phi)\rangle = \sum_{i=-n}^{n} W(e_i)S_i + \frac{1}{4}\sum_{i=-n}^{n} S_i^2 - \sum_{m\geq 3}(m-2)\mathcal{F}^{(m)}_{\chi=2} \ . \tag{3.37}$$

It is clear from the above equation that, we are close to having the free energy $\mathcal{F}_{\chi=2}$ if we can compute the expectation value $g_s\langle TrW(\Phi)\rangle$. In order to compute these expectation values, we use the following matrix model loop equation [13] :

$$w^2(x) - 2W'(x)w(x) + f_{2n}(x) = 0 \ , \tag{3.38}$$

where $w(x)$ is a resolvent of the matrix model and $f_{2n}(x)$ is an even polynomial of order $2n$ which can be chosen to be

$$f_{2n}(x) = 2W'(x)\sum_{i=-n}^{n} \frac{\tilde{S}_j}{x - e_j} \ , \tag{3.39}$$

where $\tilde{S}_j = \tilde{S}_{-j}$. The $n+1$ coefficients of the function $f_{2n}(x)$ can be related to the glueball superfields by computing the following period integral.

$$S_i = \frac{1}{2\pi i}\oint_{A_i} w(x)dx = \tilde{S}_i + \sum_{\substack{p=0 \\ m\geq 2}}^{m} \frac{(2m-3)!!}{p!(m-p)!}\frac{\tilde{S}_i^p}{(m+p-2)!}\frac{\partial^{m+p-2}}{\partial x^{m+p-2}}\frac{1}{g_{2n+2}^{m-1}R_i(x)^{m-1}}\left(\sum_{j\neq i}\frac{\tilde{S}_j}{x-e_j}\right)^{m-p}\Bigg|_{x=e_i} \tag{3.40}$$

where A_i denote the cycle enclosing the branch point centered in point e_i of the spectral curve associated with the matrix model and $R_i(x) = \prod_{j\neq i}(x - e_j)$. Also note that $S_i = S_{-i}$. Using the resolvent $\omega(x)$, the expectation value $g_s\langle TrW(\Phi)\rangle$ can be calculated from the following contour integration:

$$g_s\langle TrW(\Phi)\rangle = \frac{1}{2\pi i}\oint_A W(x)w(x)dx = \sum_{i=-n}^{n} \tilde{S}_i W(e_i) + \tag{3.41}$$

$$\sum_{i=-n}^{n}\sum_{\substack{p=0 \\ m\geq 2}}^{m} \frac{(2m-3)!!}{p!(m-p)!}\frac{\tilde{S}_i^p}{(m+p-2)!}\frac{\partial^{m+p-2}}{\partial x^{m+p-2}}\frac{W(x)}{g_{2n+2}^{m-1}R_i(x)^{m-1}}\left(\sum_{j\neq i}\frac{\tilde{S}_j}{x-e_j}\right)^{m-p}\Bigg|_{x=e_i}$$

Here the contour $A = \sum_{i=-n}^{n} A_i$. One can use eqn.(3.40) to write the above expression in terms of S_i instead of $\tilde{S}_i$. The resulting relation can be expressed in the form of eqn.(3.37). And the comparison with eqn.(3.37) gives the polynomials $\mathcal{F}^{(m)}_{\chi=2}$. As an example, for $m = 3$ we get

$$g_{2n+2}\mathcal{F}^{(3)}_{\chi=2} = -\frac{1}{2}\sum_{i=-n}^{n}\sum_{j\neq i}\sum_{k\neq i}\frac{S_iS_jS_k}{R_ie_{ij}e_{ik}} + \frac{1}{2}\sum_{i=-n}^{n}\sum_{j\neq i}\sum_{k\neq i}\frac{S_i^2S_j}{R_ie_{ij}e_{ik}} + \frac{1}{4}\sum_{i=-n}^{n}\sum_{j\neq i}\frac{S_i^2S_j}{R_ie_{ij}^2}$$
$$+\frac{1}{16}\sum_{i=-n}^{n}\sum_{j\neq i}\sum_{k\neq i}\frac{S_i^3}{R_ie_{ij}e_{ik}} - \frac{1}{6}\sum_{i=-n}^{n}\frac{S_i^3}{R_i}\left(\sum_{j\neq i}\frac{1}{e_{ij}}\right)^2 , \qquad (3.42)$$

where $e_{ij} = e_i - e_j$ and $R_i = \prod_{j\neq i}(e_i - e_j)$. This result matches with the one given in [25], where it has been written by using the relation between free energies of $U(N)$ and $SO(N)$ gauge theories.

3.2.2 Contribution of $\mathcal{F}_{\chi=1}$

For the computation of matter contribution, we incorporate the fact of broken gauge group in eqn.(3.23) as follows,

$$\mathcal{F}_{\chi=1} = g_s\sum_{I=1}^{N_f}\langle Trlog(\Phi + m_I)\rangle = \sum_{I=1}^{N_f}\sum_{i=-n}^{n} S_i log(e_i + m_I) + \sum_{m\geq 2}\mathcal{F}^{(m)}_{\chi=1} \qquad (3.43)$$

where $\mathcal{F}^{(m)}_{\chi=1}$ are polynomials in S_i of order m. We obtain $\mathcal{F}_{\chi=1}$ by evaluating the expectation value of $log(x + m_I)$.

$$\mathcal{F}_{\chi=1} = \sum_{I=1}^{N_f}\frac{1}{2\pi i}\oint_A log(x+m_I)w(x)dx = \sum_{I=1}^{N_f}\sum_{i=-n}^{n}\tilde{S}_i log(e_i + m_I) + \qquad (3.44)$$
$$\sum_{I=1}^{N_f}\sum_{i=-n}^{n}\sum_{\substack{p=0\\ m\geq 2}}^{m}\frac{(2m-3)!!}{p!(m-p)!}\frac{\tilde{S}_i^p}{(m+p-2)!}\frac{\partial^{m+p-2}}{\partial x^{m+p-2}}\frac{log(x+m_I)}{g_{2n+2}^{m-1}R_i(x)^{m-1}}\left(\sum_{j\neq i}\frac{\tilde{S}_j}{x-e_j}\right)^{m-p}\Bigg|_{x=e_i}$$

This result when expressed in terms of S_i, can be compared with eqn.(3.43) to get $\mathcal{F}^{(m)}_{\chi=1}$. For $m = 2$, the expression for $\mathcal{F}^{(2)}_{\chi=1}$ is

$$g_{2n+2}\mathcal{F}^{(2)}_{\chi=1} = \sum_{I=1}^{N_f}\sum_{i=-n}^{n}\left(\frac{S_i}{e_{iI}R_i}\sum_{j\neq i}\frac{S_j}{e_{ij}} - \frac{S_i^2}{4e_{iI}^2R_i} - \frac{S_i^2R_i'}{2e_{iI}R_i^2}\right) \qquad (3.45)$$

where $e_{iI} = e_i + m_I$. This result agrees with [25]. For $m = 3$, $\mathcal{F}^{(3)}_{\chi=1}$ takes the following form:

$$g_{2n+2}^2\mathcal{F}^{(3)}_{\chi=1} = \sum_{I=1}^{N_f}\sum_{i=-n}^{n}\frac{S_i^3}{e_{iI}R_i}\left[-\frac{1}{8e_{iI}^3R_i} - \frac{R_i'}{3e_{iI}^2R_i^2} + \frac{R_i''}{8e_{iI}R_i^2} - \frac{R_i'^2}{2e_{iI}R_i^3} - \frac{R_i'''}{6R_i^2}\right.$$

$$
\begin{aligned}
&+ \frac{5}{4}\frac{R_i' R_i''}{R_i^3} - \frac{3}{2}\frac{R_i'^3}{R_i^4} - \frac{1}{2}\sum_{j\neq i}\frac{1}{R_j e_{ij}^3}\Bigg] + \sum_{I=1}^{N_f}\sum_{i=-n}^{n}\frac{S_i^2}{e_{iI}R_i}\Bigg[\frac{1}{2e_{iI}^2 R_i}\sum_{j\neq i}\frac{S_j}{e_{ij}} \\
&+ \frac{R_i'}{e_{iI}R_i^2}\sum_{j\neq i}\frac{S_j}{e_{ij}} + \frac{1}{4e_{iI}R_i}\sum_{j\neq i}\frac{S_j}{e_{ij}^2} - \frac{5}{4}\frac{R_i''}{R_i^2}\sum_{j\neq i}\frac{S_j}{e_{ij}} + 3\frac{R_i'^2}{R_i^3}\sum_{j\neq i}\frac{S_j}{e_{ij}} \\
&+ 2\frac{R_i'}{R_i^2}\sum_{j\neq i}\frac{S_j}{e_{ij}^2} - \sum_{j\neq i}\frac{S_j R_j'}{R_j^2 e_{ij}^2} + \sum_{j\neq i}\frac{S_j}{R_j e_{ij}^3} + \frac{3}{2}\frac{1}{R_i}\sum_{j\neq i}\frac{S_j}{e_{ij}^3} + \sum_{\substack{j\neq i\\ k\neq i,j}}\frac{S_k}{R_j e_{ij}^2 e_{jk}}\Bigg] \\
&+ \sum_{I=1}^{N_f}\sum_{i=-n}^{n}\frac{S_i}{e_{iI}R_i}\times\Bigg[\sum_{\substack{j\neq i\\ k\neq j}}\frac{S_j S_k R_j'}{R_j^2 e_{ij}e_{jk}} + \sum_{\substack{j\neq i\\ k\neq j}}\frac{S_j S_k}{R_j e_{ij} e_{jk}^2} - \frac{1}{2}\frac{1}{e_{iI}R_i}\sum_{\substack{j\neq i\\ k\neq j}}\frac{S_j S_k}{e_{ij}e_{ik}} \\
&- \frac{3}{2}\frac{R_i'}{R_i^2}\sum_{\substack{j\neq i\\ k\neq j}}\frac{S_j S_k}{e_{ij}e_{ik}} - 2\frac{1}{R_i}\sum_{\substack{j\neq i\\ k\neq j}}\frac{S_j S_k}{e_{ij}e_{ik}^2} - \frac{1}{2}\sum_{\substack{j\neq i\\ k,l\neq i,j}}\frac{1}{R_j e_{ij}}\frac{S_k S_l}{e_{jk}e_{jl}} \\
&+ \frac{1}{4}\sum_{j\neq i}\frac{S_j^2 R_j''}{R_j^2 e_{ij}} - \frac{1}{2}\sum_{j\neq i}\frac{S_j^2 R_j'^2}{R_j^3 e_{ij}}\Bigg]
\end{aligned}
\tag{3.46}
$$

In principle, the above computation can be done to any order m. It is important to realize the power of assimilating Dijkgraaf-Vafa conjecture and the connections to factorization of Seiberg-Witten curves which led to such precise determination of $SO(N_c)$ effective superpotential for arbitrary polynomials of tree level superpotential at generic point in the classical moduli space (both unbroken and broken gauge group). In order to make sure that the results are consistent, we need to compare with other approaches.

In the next section, we compare the results with explicit answers obtained from geometric approach of dualities.

4 Geometric Engineering and Effective SO Superpotential

We will briefly recapitulate geometric dualities leading to the computation of SO superpotential.

4.1 Geometric Transition

Consider type IIB String theory compactified on an orientifold of a resolved Calabi-Yau geometry whose singular limit is given by eqn.(1.1). For description of SO gauge group, $W(x)$ (1.1) must be even functions of x. Further, $W'(x) = 0$ determines the eigenvalues of Φ which can be $0, \pm i a_i' s$.

We are interested in $\mathcal{N} = 1$ $SO(N_c)$ supersymmetric gauge theory in four dimensions. This can be realized by wrapping N_c D5branes on $\mathbf{RP}^2$ of the orientifolded

resolved geometry- i.e., we place all the N_c branes at $x = 0$ where eigenvalues of Φ are zero. Invoking large N duality [2,3,11], the supersymmetric gauge theory is dual to IIB string theory on a deformed Calabi-Yau geometry with fluxes. The deformed geometry is described by

$$k \equiv W'(x)^2 + f_{2n}(x) + y^2 + z^2 + w^2 = 0\,, \tag{4.1}$$

where $f_{2n}(x)$ is a n degree polynomial in x^2. The three-cycles in this geometry can be given in terms of basis cycles $A_i, B_i \in H_3(M, \mathbf{Z})$ $(i = 1, 2, \ldots 2n + 1)$ satisfying symplectic pairing

$$(A_i, B_j) = -(B_j, A_i) = \delta_{ij}\ ,\quad (A_i, A_j) = (B_i, B_j) = 0\ .$$

Here the pairing (A, B) of three-cycles A, B is defined as the intersection number. For the deformed Calabi-Yau (4.1), these three-cycles are constructed as $\mathbf{P}^1$ fibration over the line segments between two critical points $x = 0^+, 0^-, \pm i a_i^+, \pm i a_i^- \ldots$ of $W'(x)^2 + f_{2n}(x)$ in x-plane. In particular, A_0 cycle corresponds to $\mathbf{P}^1$ fibration over the line segment $0^- < x < 0^+$ and A_i's to be fibration over the line segments $i a_i^- < x < i a_i^+$. The three-cycles $B_0(B_i' s)$ are non-compact and are given by fibrations over line segments between $0 < x < \Lambda_0 (i a_i^+ < x < i\Lambda_0)$ where Λ_0 is a cut-off. The deformed geometry (4.1) has $\mathbf{Z}_2$ symmetry and hence we can restrict the discussion to the upper half of x-plane. The holomorphic three-form Ω for the deformed geometry (4.1) is give by

$$\Omega = 2\frac{dx \wedge dy \wedge dz}{\partial k/\partial \omega}\ . \tag{4.2}$$

The periods S_i and the dual periods Π_i for this deformed geometry are

$$S_i = \int_{A_i} \Omega\ ,\quad \Pi_i = \int_{B_i} \Omega\ .$$

The dual periods in terms of prepotential $\mathcal{F}(S_i)$ is $\Pi_i = \partial\mathcal{F}/\partial S_i$. Using the fact that these three cycles can be seen as $\mathbf{P}^1$ fibrations over appropriate segments in the x-plane, the periods can be rewritten as integral over a one-form ω in the x-plane. That is, $S_0 = 1/(2\pi i)\int_{0^-}^{0^+}\omega\ ,\ \ \Pi_0 = 1/(2\pi i)\int_{0^+}^{\Lambda_0}\omega\ , \ldots$ where the one-form ω is given by

$$\omega = 2dx\left(W'(x)^2 + f_{2n}(x)\right)^{\frac{1}{2}}\ . \tag{4.3}$$

The effective superpotential W^0_{eff} (recall the suffix 0 denotes the contribution from adjoint matter field Φ) can be obtained as follows

$$-\frac{1}{2\pi i}W^0_{eff} = \int \Omega \wedge (H_R + \tau H_{NS})\,, \tag{4.4}$$

where τ is the complexified coupling constant of type IIB strings, the H_R and H_{NS} denotes the RR-three form and NS-NS three-form field strengths. Inclusion of matter

in fundamental representations in the geometric framework corresponds to placing D5 branes at locations $x = m_a$ where m_a's are the masses of N_f fundamental flavors. These locations are not the zeros of $W'(x) = 0$. The fundamental matter contribution to the effective potential is given by [28]:

$$W_{eff}^{flav} \equiv \mathcal{F}_{\chi=1} = \frac{1}{2}\sum_{a=1}^{N_f}\int_{m_a}^{\Lambda_0} \omega \ . \tag{4.5}$$

For simplicity, we will confine to the classical solution $\Phi = 0$ with N_c D5-branes at $x = 0$. The corresponding dual theory will require RR-flux over A_0 cycle alone and a non-zero period $S_0 \equiv S$. The effective superpotential W_{eff}^0 (4.4) in terms of $\chi = 2$ part of matrix model free energy $\mathcal{F}(S)$ will be

$$W_{eff}^0 = \left(\frac{N_c}{2} - 1\right)\frac{\partial \mathcal{F}(S)}{\partial S} = \left(\frac{N_c}{2} - 1\right)\int_{0^+}^{\Lambda_0} w dx \tag{4.6}$$

It is important to work out explicitly these formal integrals for specific potentials and compare with our closed form expression obtained for arbitrary potentials in subsection 3.1.

4.2 Effective Superpotential for Sixth Order Potential

In this subsection we consider the $\mathcal{N} = 1$ $SO(N_c)$ gauge theory with fundamental matter and the following tree level superpotential:

$$W_{tree}(\Phi) = \frac{m}{2}Tr\Phi^2 + \frac{g}{4}Tr\Phi^4 + \frac{\lambda}{6}Tr\Phi^6 \,. \tag{4.7}$$

The geometry corresponding to this gauge theory is given by

$$W'(x)^2 + f_4(x) + y^2 + z^2 + w^2 = 0\,, \tag{4.8}$$

where $f_4(x)$ is an even polynomial of degree 4.
We concentrate on the special classical vacuum $\Phi = 0$, which is sometimes called as one cut solution in the context of matrix models [26]. We require the critical points of $W'(x)^2 + f_4(x)$ to be $0^+, 0^-$. This is achieved by the following one form:

$$\omega = 2\sqrt{W'(x)^2 + f_4(x)}dx = 2\lambda(x^2 + a)(x^2 + b)\sqrt{x^2 - 4\mu^2}dx \ , \tag{4.9}$$

where $0^{\pm} = \pm 2\mu$. Also a and b are related to the couplings of the tree level potential in the following way,

$$(a + b) = \frac{g}{\lambda} + 2\mu^2 \ ,$$
$$ab = \frac{m}{\lambda} + 2\frac{g}{\lambda}\mu^2 + 6\mu^4 \ .$$

The period integral can be computed from

$$S = \frac{1}{2\pi i}\int_{-2\mu}^{2\mu} \omega dx \ . \tag{4.10}$$

For the sixth order W_{tree}, it is explicitly given by

$$S = 2m\mu^2 + 6g\mu^4 + 20\lambda\mu^6 \ . \tag{4.11}$$

For the given one-form, the $\chi = 2$ contribution to the effective superpotential is

$$W^0_{eff} = \left(\frac{N_c}{2} - 1\right)\frac{\partial \mathcal{F}(S)}{\partial S} = \left(\frac{N_c}{2} - 1\right)\int_{2\mu}^{\Lambda_0} \omega dx \ . \tag{4.12}$$

After taking the limit $\Lambda_0 \to \infty$ and ignoring the Λ_0 dependent terms, the above equation leads to

$$W^0_{eff} = (N_c - 2)\left[S\ log(2\mu) - m\mu^2 - \frac{3}{2}g\mu^4 - \frac{10}{3}\lambda\mu^6\right] \ . \tag{4.13}$$

All higher powers of μ vanish. Substitution of S from eqn.(4.11) in W^0_{eff} obtained from factorization of Sieberg-Witten curve given by eqn.(3.19), agrees with the above result. The effective superpotential that comes from the contribution of flavors (4.5) is

$$\begin{aligned} W^{flavor}_{eff} \ &= -\sum_{I=1}^{N_f}\left[m_I\sqrt{m_I^2 - 4\mu^2}\left(\frac{1}{2}m + \frac{1}{4}gm_I^2 + \frac{1}{6}\lambda m_I^4 + \mu^2\left(\frac{g}{2} + \frac{1}{3}\lambda m_I^2 + \lambda\mu^2\right)\right)\right. \\ &+ \ \mu^2\left(m + \frac{3}{2}g\mu^2 + \frac{10}{3}\lambda\mu^4\right) + 2\mu^2\left(m + 3g\mu^2 + 10\lambda\mu^4\right)log(2\Lambda_0) \\ &- \ \left. 2\mu^2\left(m + 3g\mu^2 + 10\lambda\mu^4\right)log\left(m_I + \sqrt{m_I^2 - 4\mu^2}\right)\right] \ . \end{aligned} \tag{4.14}$$

In obtaining the above result, we take the limit $\Lambda_0 \to \infty$ and ignore the Λ_0 dependent terms. Substituting for S in terms of μ^2 (4.11) in eqn. (3.32), the result agrees with the above expression.

Substitution of $\lambda = 0$ in the above equation, we get flavor contribution of the $SO(N_c)$ gauge theory with quartic tree level superpotential.

$$\begin{aligned} W^{flavor}_{eff} \ &= -\sum_{I=1}^{N_f}\left[m_I\sqrt{m_I^2 - 4\mu^2}\left(\frac{1}{2}m + \frac{1}{4}gm_I^2 + \frac{g}{2}\mu^2\right) + \mu^2\left(m + \frac{3}{2}g\mu^2\right)\right. \\ &+ \ \left. 2\mu^2\left(m + 3g\mu^2\right)log(2\Lambda_0) - 2\mu^2\left(m + 3g\mu^2\right)log\left(m_I + \sqrt{m_I^2 - 4\mu^2}\right)\right] \ . \end{aligned} \tag{4.15}$$

The corresponding S is given by

$$S = 2m\mu^2 + 6g\mu^4 \ , \tag{4.16}$$

which is quadratic in μ^2 and can be solved to give the roots. Discarding the negative root, we get

$$\mu^2 = -\frac{m}{6g} + \frac{m}{6g}\sqrt{1 + \frac{6gS}{m^2}} \ . \tag{4.17}$$

Substituting μ^2 and rewriting in powers of S agrees with our expansion (3.31). If we take $g \to 0$ limit in the above equation, we get W_{eff}^{flavor} of the $SO(N_c)$ gauge theory with quadratic tree level superpotential.

$$W_{eff}^{flavor} = -\sum_{I=1}^{N_f}\left[\frac{S}{2} + \frac{M{m_I}^2}{2}\sqrt{1 - \frac{2S}{M{m_I}^2}} + Slog\left(\frac{\Lambda_0}{m_I}\right) - Slog\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{2S}{M{m_I}^2}}\right)\right] . \tag{4.18}$$

If we replace M by $M'/2$, we get the Affleck-Dine-Seiberg $SU(N)$ superpotential [36]. Expanding the above equation in powers of S agrees with eqn. (3.30).

Though we have considered in detail the sixth order potential, it is straightforward to obtain the effective superpotential for any polynomial potential. So far, we have discussed the results for unbroken gauge group.

It will be interesting to verify the results in section 3.2 for broken gauge group also. Some work in this direction has already been reported in Ref. [37] for quartic potential without matter. Even though a formal expressions can be written in integral form for a general polynomial potential, we still have to work out the results in a certain limit to compare with the answers in section 3.2. We hope to report on these aspects in future.

5 Summary and Discussion

In this paper, we have derived $SO(N_c)$ effective superpotential for the supersymmetric theory with N_f fundamental flavors (3.33). Using Dijkgraaf-Vafa conjecture and also the Sieberg-Witten factorization, we have obtained the effective superpotential for a most general tree level potential $W_{tree}(\Phi^2)$. We have shown agreement with the results from the geometric considerations of superstring dualities for a sixth order tree level polynomial potential. We hope to report the explicit computation within geometric framework for the broken gauge group in future.

Though we have concentrated on the SO gauge group, it appears that the fundamental matter contribution to the Sp (symplectic) effective superpotential will be identical ($\mathcal{F}_{\chi=1}$). However, one has to elaborately perform the derivation as done for SO group. The effective potential in the absence of matter is well-studied from various approaches which leads to the replacement of factor $N_c - 2$ in eqn.(3.33) by $N_c + 2$ to get W_{eff}^0 for $Sp(N_c)$ gauge group.

Within supersymmetric theories, the effective superpotentials for different regimes like $N_f = N_c$ or $N_f < N_c$ or $N_f > N_c$ could be addressed [38]. Some of these issues have been considered within the matrix model approach in [22, 39].

We have confined to a specific form of tree level potential which breaks $\mathcal{N}=2$ to $\mathcal{N}=1$. It will be interesting to look at other tree potentials involving more than one adjoint matter. We hope to report on these issues elsewhere.

Acknowledgments
PB would like to thank CSIR for the grant. The work of PR is supported by Department of Science and Technology grant under " SERC FAST TRACK Scheme for Young Scientists".

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